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2005 J. Phys. A: Math. Gen. 38 5831

(http://iopscience.iop.org/0305-4470/38/25/N01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 5831–5835 doi:10.1088/0305-4470/38/25/N01

COMMENT

Comment on 'Generalization of the Darboux transformation and generalized harmonic oscillators'

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Received 26 January 2005
Published 8 June 2005
Online at stacks.iop.org/JPhysA/38/5831

Abstract

The authors Song and Klauder (2003 *J. Phys. A: Math. Gen.* **36** 8673–84) present a generalized Darboux transformation, applicable to Hamiltonians with linear terms in the momentum. We show here that this generalized Darboux transformation is just the standard Darboux transformation in different coordinates.

PACS numbers: 03.65.Ge, 03.65.Ca

1. Comment on the paper [4]

In [4], the following Hamiltonian is considered:

$$H_0 = \frac{p^2}{2m} + (Rp + pR) + V_0, \tag{1}$$

where $R = R(x,t)$ is arbitrary, the mass $m = m(t)$ depends on time and $V_0 = V_0(x,t)$ is the potential. The Hamiltonian (1) corresponds to the following time-dependent Schrödinger equation (TDSE):

$$\mathrm{i}\Psi_t - H_0\Psi = 0 \quad \Leftrightarrow \quad \mathrm{i}\Psi_t + \frac{1}{2m}\Psi_{xx} + 2\mathrm{i}R\Psi_x + (\mathrm{i}R_x - V_0)\Psi = 0. \tag{2}$$

Here, the indices denote partial derivatives. It is well known [3] that the term $\sim\Psi_x$ can be removed by changing the dependent coordinate Ψ as

$$\Psi = \exp\left(-2\mathrm{i}m \int R\,\mathrm{d}x\right)\Phi, \tag{3}$$

which changes (2) into an equation for Φ:

$$\mathrm{i}\Phi_t + \frac{1}{2m}\Phi_{xx} + \left(2\left(m\int R\,\mathrm{d}x\right)_t + 2mR^2 - V_0\right)\Phi = 0. \tag{4}$$

This equation has the standard form of a TDSE (without terms in Ψ_x), to which the standard first-order Darboux transformation is applicable. As is well known [1, 2], the n-fold iteration of this first-order Darboux transformation has the form

$$\Phi = \prod_{j=0}^{n} \left(\frac{\partial}{\partial x} - \frac{W_{j-1}}{W_j} \left(\frac{W_j}{W_{j-1}} \right)_x \right) \chi,$$

where χ is a solution of the transformed TDSE (given below) and W_j is the following Wronski determinant:

$$W_j = \begin{vmatrix} u_0 & u_1 & \cdots & u_{j-1} \\ (u_0)_x & (u_1)_x & \cdots & (u_{j-1})_x \\ (u_0)_{xx} & (u_1)_{xx} & \cdots & (u_{j-1})_{xx} \\ \vdots & \vdots & \vdots & \vdots \\ (u_0)_{(x,j-1)} & (u_1)_{(x,j-1)} & \cdots & (u_{j-1})_{(x,j-1)} \end{vmatrix}.$$

The index (x, j) stands for the jth partial derivative with respect to x and the functions u_j are solutions of the TDSE (4). The transformed TDSE for χ reads

$$\mathrm{i}\chi_t + \frac{1}{2m}\chi_{xx} + \left(2\left(m \int R\,\mathrm{d}x \right)_t + 2mR^2 - V_0 + \frac{1}{m}(\log(W_n))_{xx} \right) \chi = 0. \quad (5)$$

Now, after having performed the Darboux transformation, we invert our change of coordinates (3) for getting back to the original form of the TDSE (2). The inverted change of coordinate reads obviously

$$\chi = \exp\left(2\mathrm{i}m \int R\,\mathrm{d}x \right) \Theta, \quad (6)$$

which on substitution into equation (5) gives

$$\mathrm{i}\Theta_t + \frac{1}{2m}\Theta_{xx} + 2\mathrm{i}R\Theta_x + \left(\mathrm{i}R_x - V_0 + \frac{1}{m}(\log(W_n))_{xx} \right) \Theta = 0. \quad (7)$$

Now we show that equation (7) and its solution are the same as the ones obtained in [4]. Let us start with the equation. We must express the entries of the Wronskian W_n in terms of solutions of equation (2) (instead of (4), as it is now). We just have to apply the same change of coordinate as in (6) to all entries of W_j, which clearly transforms them into solutions of (2). Let us abbreviate the change of coordinates by

$$T = \exp\left(2\mathrm{i}m \int R\,\mathrm{d}x \right), \quad (8)$$

and let the functions v_j be solutions of equation (2). Then the u_j and the v_j are related to each other by the change of coordinate T as

$$u_j = Tv_j.$$

We get for the Wronskian W_n:

$$W_n = \begin{vmatrix} Tv_0 & Tv_1 & \cdots & Tv_{n-1} \\ (Tv_0)_x & (Tv_1)_x & \cdots & (Tv_{n-1})_x \\ (Tv_0)_{xx} & (Tv_1)_{xx} & \cdots & (Tv_{n-1})_{xx} \\ \vdots & \vdots & \vdots & \vdots \\ (Tv_0)_{(x,n-1)} & (Tv_1)_{(x,n-1)} & \cdots & (Tv_{n-1})_{(x,n-1)} \end{vmatrix}.$$

Now we extract the term $\sim T$ from each derivative:

$$(Tv_j)_{(x,k)} = T(v_j)_{(x,k)} + r_{j,k},$$

where $r_{j,k} = r_{j,k}(x,t)$ stands for the remaining terms of the derivative. We get

$$W_n = \begin{vmatrix} Tv_0 & Tv_1 & \cdots & Tv_{n-1} \\ T(v_0)_x + r_{0,1} & T(v_1)_x + r_{1,1} & \cdots & T(v_{n-1})_x + r_{n-1,1} \\ T(v_0)_{xx} + r_{0,2} & T(v_1)_{xx} + r_{1,2} & \cdots & T(v_{n-1})_{xx} + r_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots \\ T(v_0)_{(x,n-1)} + r_{0,n-1} & T(v_1)_{(x,n-1)} + r_{1,n-1} & \cdots & T(v_{n-1})_{(x,n-1)} + r_{n-1,n-1} \end{vmatrix}.$$

Next, we pull out the factor T from each row, leaving the determinant as

$$W_n = T^n \begin{vmatrix} v_0 & v_1 & \cdots & v_{n-1} \\ (v_0)_x + \frac{1}{T}r_{0,1} & (v_1)_x + \frac{1}{T}r_{1,1} & \cdots & (v_{n-1})_x + \frac{1}{T}r_{n-1,1} \\ (v_0)_{xx} + \frac{1}{T}r_{0,2} & (v_1)_{xx} + \frac{1}{T}r_{1,2} & \cdots & (v_{n-1})_{xx} + \frac{1}{T}r_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots \\ (v_0)_{(x,n-1)} + \frac{1}{T}r_{0,n-1} & (v_1)_{(x,n-1)} + \frac{1}{T}r_{1,n-1} & \cdots & (v_{n-1})_{(x,n-1)} + \frac{1}{T}r_{n-1,n-1} \end{vmatrix}.$$

Finally, we perform some elementary operations on the latter determinant. We have

$$\frac{1}{T}r_{j,1} = \frac{1}{T}T_x v_j.$$

Now, on multiplying the first row by T_x/T and subtracting it from the second row, the Wronskian takes the form

$$W_n = T^n \begin{vmatrix} v_0 & v_1 & \cdots & v_{n-1} \\ (v_0)_x & (v_1)_x & \cdots & (v_{n-1})_x \\ (v_0)_{xx} + \frac{1}{T}r_{0,2} & (v_1)_{xx} + \frac{1}{T}r_{1,2} & \cdots & (v_{n-1})_{xx} + \frac{1}{T}r_{n-1,2} \\ \vdots & \vdots & \vdots & \vdots \\ (v_0)_{(x,n-1)} + \frac{1}{T}r_{0,n-1} & (v_1)_{(x,n-1)} + \frac{1}{T}r_{1,n-1} & \cdots & (v_{n-1})_{(x,n-1)} + \frac{1}{T}r_{n-1,n-1} \end{vmatrix}.$$

Now we deal with the third row. We have

$$\frac{1}{T}r_{j,2} = 2\frac{1}{T}T_x(v_j)_x + \frac{1}{T}T_{xx}v_j.$$

Hence, on multiplying the first row by T_{xx}/T, multiplying the second row by $2T_x/T$, adding these rows and subtracting the result from the third row, we obtain

$$W_n = T^n \begin{vmatrix} v_0 & v_1 & \cdots & v_{n-1} \\ (v_0)_x & (v_1)_x & \cdots & (v_{n-1})_x \\ (v_0)_{xx} & (v_1)_{xx} & \cdots & (v_{n-1})_{xx} \\ \vdots & \vdots & \vdots & \vdots \\ (v_0)_{(x,n-1)} + \frac{1}{T}r_{0,n-1} & (v_1)_{(x,n-1)} + \frac{1}{T}r_{1,n-1} & \cdots & (v_{n-1})_{(x,n-1)} + \frac{1}{T}r_{n-1,n-1} \end{vmatrix}.$$

After proceeding in a similar manner with the remaining rows, we come to

$$W_n = T^n \begin{vmatrix} v_0 & v_1 & \cdots & v_{n-1} \\ (v_0)_x & (v_1)_x & \cdots & (v_{n-1})_x \\ (v_0)_{xx} & (v_1)_{xx} & \cdots & (v_{n-1})_{xx} \\ \vdots & \vdots & \vdots & \vdots \\ (v_0)_{(x,n-1)} & (v_1)_{(x,n-1)} & \cdots & (v_{n-1})_{(x,n-1)} \end{vmatrix}. \quad (9)$$

We abbreviate the Wronskian on the right-hand side with $\hat{W}_n$. Thus, on inserting the result (9) together with the explicit form of T as given in (8) into the potential term of equation (7), we get

$$\begin{aligned}\frac{1}{m}(\log(W_n))_{xx} &= \frac{1}{m}(\log(T^n\hat{W}_n))_{xx}\\ &= \frac{1}{m}\left(\log\left(\exp\left(2\mathrm{i}nm\int R\,\mathrm{d}x\right)\hat{W}_n\right)\right)_{xx}\\ &= \frac{1}{m}\left(2\mathrm{i}nm\int R\,\mathrm{d}x + \log(\hat{W}_n)\right)_{xx}\\ &= 2\mathrm{i}nR_x + \frac{1}{m}(\log(\hat{W}_n))_{xx}.\end{aligned}$$

Finally, we insert the latter into our equation (7), yielding

$$\mathrm{i}\Theta_t + \frac{1}{2m}\Theta_{xx} + 2\mathrm{i}R\Theta_x + \left(\mathrm{i}R_x - V_0 + 2\mathrm{i}nR_x + \frac{1}{m}(\log(\hat{W}_n))_{xx}\right)\Theta = 0.$$

This is exactly the result for the transformed equation in [4].

It remains to show that the solution of equation (7) is the same as the solution given in [4]. According to [1], the solution of (5) reads

$$\chi = \frac{W_{n,\Phi}}{W_n},$$

where

$$W_{n,\Phi} = \begin{vmatrix} u_0 & u_1 & \cdots & u_{n-1} & \Phi\\ (u_0)_x & (u_1)_x & \cdots & (u_{n-1})_x & \Phi_x\\ (u_0)_{xx} & (u_1)_{xx} & \cdots & (u_{n-1})_{xx} & \Phi_{xx}\\ \vdots & \vdots & \vdots & \vdots & \\ (u_0)_{(x,n-1)} & (u_1)_{(x,n-1)} & \cdots & (u_{n-1})_{(x,n-1)} & \Phi_{(x,n-1)}\end{vmatrix}.$$

Thus, the solution of (7) is given by (6), that is

$$\begin{aligned}\Theta &= \frac{1}{T}\chi\\ &= \frac{1}{T}\frac{W_{n,\Phi}}{W_n}.\end{aligned} \tag{10}$$

Clearly, the entries of these Wronskians are solutions of equation (4). As before, we express them through solutions of equation (2) by multiplication of each entry with a factor T. On performing the same operations as in the last paragraph we obtain

$$W_{n,\Phi} = T^{n+1}\hat{W}_{n,\Phi} \qquad W_n = T^n\hat{W}_n.$$

These results we insert into (10) and get

$$\Theta = \frac{\hat{W}_{n,\Phi}}{\hat{W}_n}.$$

This coincides with the result in [4].

In summary, we have shown that the generalized Darboux transformation in [4] is the standard Darboux transformation up to a change of the dependent coordinate.

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